Modular Integrals in Minimal Super Liouville Gravity

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Abstract The four-point integral of the minimal super Liouville gravity on the sphere is evaluated numerically. The integration procedure is based on the effective elliptic parameterization of the moduli space. The analysis is performed for a few different gravitational four-point amplitudes. The results agree with the analytic results recently obtained using the Higher super Liouville equations of motion.

1 Introduction

The continuum formulation of the noncritical string theory is equivalent to 2D quantum gravity coupled to some critical matter, i.e., the matter described by a conformal field theory $\mathcal{M}_c$. Simple reaction of conformal theories to the scaling of the metric leads to the universal form of the effective action of the generated gravity, which is called the Liouville gravity (LG) [1]. Because of the peculiarities of two-dimensional metric geometry, many technical simplifications immediately come into play. Thus, LG is perhaps the simplest example of quantum gravity, but it nevertheless shares the same basic questions of interpretation and can hence be considered useful and worth studying. The problem of choosing observables correctly and the problem of calculating the corresponding correlation functions are of primary importance in any quantum theory. The field of LG has experienced considerable progress in recent years. Recently discovered higher equations of motion (HEM) [2] in LFT have allowed reaching the four-point level in calculating the correlation functions in LG [3, 4]. The results were tested against the calculation in the framework of the relatively independent approach to 2D quantum gravity usually called the matrix models (see, e.g., [5] and the references therein). Moreover, a deeper understanding of the correspondence between these two approaches was achieved based on these results [6], although the complete picture of the relations between these techniques is still missing.

In the context of string theory applications, the construction of 2D quantum gravity in superspace is one of the most interesting questions. The first possible generalization is N=1 supersymmetry. Here, essential progress was also achieved recently. The study was motivated by a series of works concerned with super LFT in which the profound understanding of the properties of the conformal blocks (which are the basic elements of any CFT [7]) was

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achieved. This allowed "taking off" the necessity of treating the correlations involving only special degenerate [8] excitations. Another important result was the discovery of the HEM in super LFT [9]. All these results have served as a good starting point for a more profound study of super Liouville gravity (SLG). In [10], the structure of the physical fields in the Neveu–Schwarz (NS) sector of SLG was clarified, and the general expression for the n-point correlation numbers on the sphere in terms of integrals over moduli space was written explicitly. Then the super HEM in the N=1 super LFT and the analysis of the structure of the super ground ring (and its logarithmic counterpart) allowed deriving the explicit analytic expressions for the four-point correlators. Because little is now known for the supersymmetric matrix model, there are no independent results analogous to those obtained in [10]. In this situation, more checks of the validity of these results would seem desirable. This paper is devoted to directly calculating the four-point correlation numbers in SMLG.

The paper is organized as follows. To make the presentation self-contained, we collect all necessary information related to the subject in Sec. 2 and Sec. 3. The remaining part of the paper deals with evaluating the four-point integrals numerically. In Sec. 4, we consider two examples of the four-point integral. We reduce the expressions to a number of integrals over the fundamental region of the modular group, and the integrands represent the products of the various correlation functions both in the matter and in the Liouville sectors. The elliptic transformation is the subject of Sec. 5. The numerical results are presented in Sec. 6. Some useful details omitted from the main text are given in Appendix A.

2 Minimal Super Liouville Gravity

In the framework of the so-called DDK approach [11–13], SLG is represented as a tensor product of superconformal matter (SM), super Liouville, and super ghost systems

$$A_{\mathrm{SLG}} = A_{\mathrm{SM}} + A_{\mathrm{SL}} + A_{\mathrm{SG}} \tag{1}$$

with the interaction via the relation for the central charge parameters

$$c_{\mathrm{SM}} + c_{\mathrm{SL}} + c_{\mathrm{SG}} = 0 \tag{2}$$

and also due to the construction of the physical fields.

The superconformal algebra is

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n,-m}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \\ [L_n, G_r] &= \left(\frac{1}{2}n - r\right)G_{n+r}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &r, s \in \mathbb{Z} + \frac{1}{2} \quad \text{for the NS sector,} \\ &r, s \in \mathbb{Z} \qquad\quad \text{for the R sector.} \end{aligned}$$

The SLFT central charge is

$$\widehat{c}_{\mathrm{SL}} = 1 + 2Q^2, \tag{4}$$

where the "background charge" parameter Q is related to the SLFT basic quantum parameter b

$$Q = b^{-1} + b \tag{5}$$

The fields belong to the highest-weight representations of the superconformal algebra. The basic fields of interest in this paper belong to the primary supermultiplet $(V_a, Y_a, \bar{Y}_a, W_a)$ with the bottom component V_a having the conformal dimension

$$\Delta_a^{\mathrm{L}} = \frac{a(Q-a)}{2}, \tag{6}$$

where a is a continuous (complex) parameter, and the other components of the primary supermultiplet being

$$Y_a = G_{-1/2}^{\mathrm{L}} V_a, \quad \bar{Y}_a = \bar{G}_{-1/2}^{\mathrm{L}} V_a, \quad W_a = \bar{G}_{-1/2}^{\mathrm{L}} G_{-1/2}^{\mathrm{L}} V_a. \tag{7}$$

Here and hereafter, we use the superscripts M, L, and G to specify the matter, Liouville, and ghost sectors of the superconformal generators. The generators without sector superscripts are related to the total super Virasoro algebra.

At certain special values of the parameter $a = a_{m,n}$, one singular vector appears at the level $mn/2$ in the Verma module over $V_{a_{m,n}} = V_{m,n}$ [8]. Here,

$$a_{m,n} = Q/2 - \lambda_{m,n}, \tag{8}$$

where (m, n) is a pair of positive integers ($m - n \in 2\mathbb{Z}$) and

$$\lambda_{m,n} = \frac{mb^{-1} + nb}{2}. \tag{9}$$

The basic super Liouville operator product expansion [14] (for the sake of brevity we write $\Delta = \Delta_{Q/2+iP}$ and $\Delta_i = \Delta_{a_i}$)

$$\begin{aligned} &V_{a_1}(x) V_{a_2}(0) = \\ &\quad \int' \frac{dP}{4\pi} (x\bar{x})^{\Delta - \Delta_1 - \Delta_2} \left(\mathbb{C}_{a_1,a_2}^{Q/2+iP} \left[V_{Q/2+iP}(0) \right]_{\mathrm{ee}} + \tilde{\mathbb{C}}_{a_1,a_2}^{Q/2+iP} \left[V_{Q/2+iP}(0) \right]_{\mathrm{oo}} \right) \end{aligned} \tag{10}$$

This OPE is continuous and involves integration over the "momentum" P. In (10), $[V_P]_{\mathrm{ee,oo}}$ denotes the contribution of the primary field V_p and its "even" and "odd" superconformal descendants to the operator product expansion. As usual, the prime on the integral indicates possible discrete terms; in this study, we consider only the region b where such extra terms do not appear and the integral can be understood literally. All other OPEs of two arbitrary local fields in SLFT can be derived from (10). The basic structure constants $\mathbb{C}_{a_1 a_2}^{Q/2+iP}$ and

$\tilde{\mathbb{C}}^{Q/2+iP}_{a_1,a_2}$ in (10) were evaluated using the bootstrap technique in [15–17] and have the explicit form (here a denotes $a_1 + a_2 + a_3$)

$$\begin{aligned}
\mathbb{C}^{Q-a_3}_{a_1 a_2} &= \left(\pi\mu\gamma\left(\frac{Qb}{2}\right) b^{1-b^2}\right)^{(Q-a)/b} \frac{\Upsilon_{\mathrm{R}}(b)\Upsilon_{\mathrm{NS}}(2a_1)\Upsilon_{\mathrm{NS}}(2a_2)\Upsilon_{\mathrm{NS}}(2a_3)}{2\Upsilon_{\mathrm{NS}}(a-Q)\Upsilon_{\mathrm{NS}}(a_{1+2-3})\Upsilon_{\mathrm{NS}}(a_{2+3-1})\Upsilon_{\mathrm{NS}}(a_{3+1-2})}, \\
\tilde{\mathbb{C}}^{Q-a_3}_{a_1 a_2} &= -\left(\pi\mu\gamma\left(\frac{Qb}{2}\right) b^{1-b^2}\right)^{(Q-a)/b} \frac{i\Upsilon_{\mathrm{R}}(b)\Upsilon_{\mathrm{NS}}(2a_1)\Upsilon_{\mathrm{NS}}(2a_2)\Upsilon_{\mathrm{NS}}(2a_3)}{\Upsilon_{\mathrm{R}}(a-Q)\Upsilon_{\mathrm{R}}(a_{1+2-3})\Upsilon_{\mathrm{R}}(a_{2+3-1})\Upsilon_{\mathrm{R}}(a_{3+1-2})},
\end{aligned} \tag{11}$$

where we use the convenient notation in [17] for the special functions

$$\begin{aligned}
\Upsilon_{\mathrm{NS}}(x) &= \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right), \\
\Upsilon_{\mathrm{R}}(x) &= \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right)
\end{aligned} \tag{12}$$

expressed in terms of the "upsilon" function Υ_b, which is the standard element in the Liouville field theory (see [18, 19]).

Because of central charge balance condition (2), the central charge of the matter sector is given in terms of the same basic parameter b:

$$\widehat{c}_{\mathrm{SM}} = 1 - 2q^2, \tag{13}$$

where $q = b^{-1} - b$. We let $(\Phi_\alpha, \chi_\alpha, \bar{\chi}_\alpha, \Psi_\alpha)$ denote the primary multiplet in the matter sector with the dimension of the bottom component Φ_a being given by

$$\Delta^{\mathrm{M}}_\alpha = \frac{\alpha(q-\alpha)}{2}. \tag{14}$$

The super ghost system (see, e.g., [20–22]) is described by the free super conformal field theory with the central charge $c_{\mathrm{SG}} = -10$. The fermionic part of the SG system involves two anticommuting fields (b, c) of spins $(2, -1)$, and the bosonic part involves two bosonic fields (β, γ) of spins $(3/2, -1/2)$. The formal fields (see [10]) of the form $\delta(\gamma(0))$ of dimension $1/2$ are essential in constructing the gravitational amplitudes.

3 Physical Fields and the Correlation Numbers

The physical fields form a space of cohomology classes with respect to the nilpotent BRST charge $\mathbb{Q}$,

$$\mathbb{Q} = \sum_m :\left[L^{\mathrm{M+L}}_m + \frac{1}{2}L^{\mathrm{g}}_m\right]c_{-m}: + \sum_r :\left[G^{\mathrm{M+L}}_r + \frac{1}{2}G^{\mathrm{g}}_r\right]\gamma_{-r}: - \frac{1}{4}c_0. \tag{15}$$

In this work, we deal with the correlators of physical fields of the two types

$$\mathbb{W}_a(z,\bar{z}) = U_a(z,\bar{z}) \cdot c(z)\bar{c}(\bar{z}) \cdot \delta(\gamma(z))\delta(\bar{\gamma}(\bar{z})), \tag{16}$$

and

$$\tilde{\mathbb{W}}_a(z,\bar{z}) = \left(\bar{G}^{\mathrm{M+L}}_{-1/2} + \frac{1}{2}\bar{G}^{\mathrm{g}}_{-1/2}\right)\left(G^{\mathrm{M+L}}_{-1/2} + \frac{1}{2}G^{\mathrm{g}}_{-1/2}\right)\mathbb{U}_a(z,\bar{z}) \cdot \bar{c}(\bar{z})c(z), \tag{17}$$

where

$$\mathbb{U}_a(z,\bar{z}) = \Phi_{a-b}(z,\bar{z})V_a(z,\bar{z}) \tag{18}$$

Here the parameter a can take generic values. The general form of the n-point correlation numbers on the sphere for these observables [10] is

$$I_n(a_1,\cdots,a_n) = \prod_{i=4}^{n}\int d^2z_i \left\langle \bar{G}_{-1/2}G_{-1/2}\mathbb{U}_{a_i}(z_i)\tilde{\mathbb{W}}_{a_3}(z_3)\,\mathbb{W}_{a_2}(z_2)\,\mathbb{W}_{a_1}(z_1)\right\rangle. \tag{19}$$

An additional "discrete" physical state arises when the representation in the matter sector is degenerate,

$$\mathbb{O}_{m,n}(z,\bar{z}) = \bar{H}_{m,n}H_{m,n}\Phi_{m,n}(z,\bar{z})V_{m,n}(z,\bar{z}). \tag{20}$$

The operators $H_{m,n}$ are composed of the super Virasoro generators and are defined uniquely modulo exact terms. Moreover, if we introduce the logarithmic counterparts of the discrete states $\mathbb{O}_{m,n}$,

$$\mathbb{O}'_{m,n} = \bar{H}_{m,n}H_{m,n}\Phi_{m,n}V'_{m,n}, \tag{21}$$

then we have the important relations [10]

$$\bar{\mathbb{Q}}\mathbb{Q}\mathbb{O}'_{m,n} = B_{m,n}\tilde{\mathbb{W}}_{m,-n} \tag{22}$$

and

$$\bar{G}_{-1/2}G_{-1/2}\mathbb{U}_{m,-n} = B^{-1}_{m,n}\bar{\partial}\partial\mathbb{O}'_{m,n} \quad \mod \mathbb{Q}, \tag{23}$$

where $B_{m,n}$ are the coefficients arising in the higher equations of motion of SLFT [9]. For four points, relation (23) allows reducing the moduli integral in general expression (19) to boundary integrals if one of the fields is degenerate, i.e., $a_i = a_{m,-n}$. The explicit result is (see [10])

$$I_4(a_{m,-n},a_1,a_2,a_3) = \kappa N(a_{m,-n})\left\{\sum_{i=1}^{3}\sum_{r,s\in(m,n)} q^{(m,n)}_{r,s}(a_i) + 2mn\lambda_{m,n}\right\}\prod_{i=1}^{3}N(a_i), \tag{24}$$

where

$$q^{(m,n)}_{r,s}(a) = |a-\lambda_{r,s}-Q/2|_{\mathrm{Re}} - \lambda_{m,n} \tag{25}$$

and the fusion set is $(m,n) = \{1-m:2:m-1, 1-n:2:n-1\}$. The coefficient is

$$\kappa = -2\mu^{-1}b^{-2}\left[\pi\mu\gamma\left(\frac{1}{2}+\frac{b^2}{2}\right)\right]^{2+b^{-2}}\gamma\left[\frac{b^{-2}}{2}-\frac{1}{2}\right], \tag{26}$$

and the "leg" factors are

$$N(a) = \left[\pi\mu\gamma\left(\frac{1}{2}+\frac{b^2}{2}\right)\right]^{-a/b}\left[\gamma\left(ab-\frac{b^2}{2}+\frac{1}{2}\right)\gamma\left(\frac{a}{b}-\frac{b^{-2}}{2}+\frac{1}{2}\right)\right]^{1/2}. \tag{27}$$

4 Direct Calculation

Here, we verify analytic result (24). The space of parameters (a_1, a_2, a_3, and also b) is rather big to present a comprehensive analysis. In what follows, we focus on the two examples where one of the fields is either $\mathbb{W}_b$ or $\mathbb{W}_{2b}$. Moreover, we restrict ourself to considering the most symmetric situation of four identical fields $\mathcal{I}_4(a) = I_4(a, a, a, a)$. In the four-point case (19) reduces to

$$\mathcal{I}_4(a) = \int d^2z \langle \bar{G}_{-1/2} G_{-1/2} \mathbb{U}_a(z) \mathbb{W}_a(0) \, \tilde{\mathbb{W}}_a(1) \, \mathbb{W}_a(\infty) \rangle. \tag{28}$$

The analytic results following from expression (24) are

$$\mathcal{I}_4(b) = \frac{\kappa}{b} N^4(b) \Sigma^{(1,1)}(b), \tag{29}$$

where

$$\Sigma^{(1,1)}(b) = |2b^2 - 1|. \tag{30}$$

For the second integral, we have

$$\mathcal{I}_4(2b) = \frac{\kappa}{b} N^4(2b) \Sigma^{(1,3)}(b) \tag{31}$$

and

$$\Sigma^{(1,3)}(b) = \frac{3}{2}\Bigg[|5b^2 - 1| + |3b^2 - 1| + |b^2 - 1| - 3b^2 - 1\Bigg]. \tag{32}$$

Let us consider the integral $\mathcal{I}_4(b)$. Taking into account that we deal with the unit operators in the matter sector in this case, we have

$$\begin{aligned} &\bar{G}_{-1/2} G_{-1/2} \mathbb{U}_b = W_b, \\ &\mathbb{W}_b = V_b \, \bar{c}c \, \delta(\bar{\gamma}) \delta(\gamma), \\ &\tilde{\mathbb{W}}_b = \left(\bar{G}^{\mathrm{M+L}}_{-1/2} + \frac{1}{2} \bar{G}^{\mathrm{g}}_{-1/2}\right)\left(G^{\mathrm{M+L}}_{-1/2} + \frac{1}{2} G^{\mathrm{g}}_{-1/2}\right) V_b \, \bar{c}c. \end{aligned} \tag{33}$$

Taking the explicit form of the correlation functions in the ghost sector into account,

$$\begin{aligned} \langle C(0) C(1) \rangle &= 0, \\ \langle C(0) C(1) C(\infty) \rangle &= 1, \\ \langle \delta(\gamma(0)) \delta(\gamma(1)) \rangle &= 1, \end{aligned} \tag{34}$$

we conclude that the only nonzero contribution comes from the term in $\tilde{\mathbb{W}}_b$, which is proportional to $\bar{c}c$,

$$\mathcal{I}_4(b) = \int d^2z \langle W_b(z) V_b(0) W_b(1) V_b(\infty) \rangle. \tag{35}$$

In the same way, we have

$$
\begin{aligned}
&\bar{G}_{-1/2}G_{-1/2}\mathbb{U}_{2b} = \bar{G}^{\mathrm{M+L}}_{-1/2}G^{\mathrm{M+L}}_{-1/2}\Phi_b V_{2b},\\
&\mathbb{W}_{2b} = \Phi_b V_{2b}\,\bar{c}c\,\delta(\bar{\gamma})\delta(\gamma),\\
&\tilde{\mathbb{W}}_{2b} = \left(\bar{G}^{\mathrm{M+L}}_{-1/2} + \frac{1}{2}\bar{G}^{\mathrm{g}}_{-1/2}\right)\left(G^{\mathrm{M+L}}_{-1/2} + \frac{1}{2}G^{\mathrm{g}}_{-1/2}\right)\Phi_b V_{2b}\,\bar{c}c,
\end{aligned}
\tag{36}
$$

for the second integral, and taking (34) into account, we obtain

$$
\begin{aligned}
\mathcal{I}_4(2b) = \int d^2z\Big(&\langle\Psi_b(z)\Phi_b(0)\Psi_b(1)\Phi_b(\infty)\rangle\langle V_{2b}(z)V_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle,\\
&+\langle\chi_b(z)\Phi_b(0)\chi_b(1)\Phi_b(\infty)\rangle\langle\bar{Y}_{2b}(z)V_{2b}(0)\bar{Y}_{2b}(1)V_{2b}(\infty)\rangle,\\
&+\langle\bar{\chi}_b(z)\Phi_b(0)\bar{\chi}_b(1)\Phi_b(\infty)\rangle\langle Y_{2b}(z)V_{2b}(0)Y_{2b}(1)V_{2b}(\infty)\rangle,\\
&+\langle\Phi_b(z)\Phi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle W_{2b}(z)V_{2b}(0)W_{2b}(1)V_{2b}(\infty)\rangle.\Big)
\end{aligned}
\tag{37}
$$

We now use the symmetry of the integrals under modular transformations to reduce the integration from the whole complex plane to the fundamental domain. The modular subgroup of projective transformations divides the complex plane into six regions. The fundamental region is defined as $\mathbf{G} = \{\mathrm{Re}\,x < 1/2;\ |1-x| < 1\}$. The other five regions are mapped to the fundamental one using one of the transformations $\mathcal{A}, \mathcal{B}, \mathcal{AB}, \mathcal{BA}, \mathcal{ABA}$, where $\mathcal{A}$: $z \to 1/z$ and $\mathcal{B}$: $z \to 1-z$. Combining the projective transformations of the fields and the corresponding change of variables in the integrals, we reduce the integration to the fundamental region. We note that the Jacobian of the transformation exactly cancels the transformation of the fields because their total conformal dimension is 1. Then,

$$
\begin{aligned}
\mathcal{I}_4(b) = 2\int_{\mathbf{G}} d^2z\Big(&\langle W_b(z)V_b(0)W_b(1)V_b(\infty)\rangle + \langle W_b(z)V_b(0)V_b(1)W_b(\infty)\rangle +\\
&+\langle W_b(z)W_b(0)V_b(1)V_b(\infty)\rangle\Big),
\end{aligned}
\tag{38}
$$

where the factor 2 in front of the integral takes the equivalent projective images into account.

The expression for the second integral is rather bulky:

$$
\begin{aligned}
\mathcal{I}_4(2b) = 2\int_{\mathbf{G}} d^2z \Bigg[\Big(& \langle \Psi_b(z)\Phi_b(0)\Psi_b(1)\Phi_b(\infty)\rangle\langle V_{2b}(z)V_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \Psi_b(z)\Psi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle V_{2b}(z)V_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \Psi_b(z)\Phi_b(0)\Phi_b(1)\Psi_b(\infty)\rangle\langle V_{2b}(z)V_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \Big) \\
& +\Big(\langle \chi_b(z)\Phi_b(0)\chi_b(1)\Phi_b(\infty)\rangle\langle \bar{Y}_{2b}(z)V_{2b}(0)\bar{Y}_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \chi_b(z)\chi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle \bar{Y}_{2b}(z)\bar{Y}_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \chi_b(z)\Phi_b(0)\Phi_b(1)\chi_b(\infty)\rangle\langle \bar{Y}_{2b}(z)V_{2b}(0)V_{2b}(1)\bar{Y}_{2b}(\infty)\rangle \Big) \\
& +\Big(\langle \bar{\chi}_b(z)\Phi_b(0)\bar{\chi}_b(1)\Phi_b(\infty)\rangle\langle Y_{2b}(z)V_{2b}(0)Y_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \bar{\chi}_b(z)\bar{\chi}_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle Y_{2b}(z)Y_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \bar{\chi}_b(z)\Phi_b(0)\Phi_b(1)\bar{\chi}_b(\infty)\rangle\langle Y_{2b}(z)V_{2b}(0)V_{2b}(1)Y_{2b}(\infty)\rangle \Big) \\
& +\Big(\langle \Phi_b(z)\Phi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle W_{2b}(z)V_{2b}(0)W_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \Phi_b(z)\Phi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle W_{2b}(z)W_{2b}(0)V_{2b}(1)V_{2b}(\infty)\rangle \\
& +\langle \Phi_b(z)\Phi_b(0)\Phi_b(1)\Phi_b(\infty)\rangle\langle W_{2b}(z)V_{2b}(0)V_{2b}(1)W_{2b}(\infty)\rangle \Big) \Bigg].
\end{aligned} \tag{39}
$$

We now use the conformal block decomposition of the correlation functions. It is useful to introduce a compact notation. For a while, we omit some arguments that are easily reconstructed in the final expressions. In the matter sector,

$$
\begin{aligned}
\langle \Phi(z)\Phi(0)\Phi(1)\Phi(\infty)\rangle &= c_k|A_k^{(0)}(z)|^2, \\
\langle \Psi(z)\Phi(0)\Psi(1)\Phi(\infty)\rangle &= c_k|A_k^{(1)}(z)|^2, \\
\langle \Psi(z)\Psi(0)\Phi(1)\Phi(\infty)\rangle &= c_k|A_k^{(2)}(z)|^2, \\
\langle \Psi(z)\Phi(0)\Phi(1)\Psi(\infty)\rangle &= c_k|A_k^{(3)}(z)|^2.
\end{aligned} \tag{40}
$$

Here, the index $k = +, 0, -$ corresponds to the three channels in the degenerate OPE of the field Φ_b (and also of its super partners), and we assume summation with respect to k. The coefficients c_k are related to the basic structure constants (see [10]):

$$c_+ = C_+^2(b) = \frac{\gamma(1/2+b^2/2)\gamma(-1/2+5b^2/2)}{\gamma(-1/2+3b^2/2)\gamma(1/2+3b^2/2)}, \tag{41}$$

$$c_0 = -\tilde{C}_0^2(b) = -1, \tag{42}$$

$$c_- = C_-^2(b) = -\frac{\gamma(2b^2)\gamma(-1/2+b^2/2)\gamma^2(1/2+b^2/2)}{b^4\gamma^3(b^2)\gamma(-1+b^2)\gamma(-1/2+3b^2/2)}. \tag{43}$$

In (40), $A_k^{(n)}$ denotes the conformal blocks appearing in the k channel for the given correlation function. Here and hereafter, the normalization is chosen such that all but the basic combinations c_k are absorbed inside the conformal blocks. In Appendix A, we recapitulate some details and explicit constructions concerning conformal blocks. In the Liouville sector,

$$\begin{aligned}
\langle V(z)V(0)V(1)V(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P)|B_l^{(0)}(P,z)|^2,\\
\langle W(z)V(0)W(1)V(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P)|B_l^{(1)}(P,z)|^2,\\
\langle W(z)W(0)V(1)V(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P)|B_l^{(2)}(P,z)|^2,\\
\langle W(z)V(0)V(1)W(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P)|B_l^{(3)}(P,z)|^2.
\end{aligned} \tag{44}$$

Because the correlation functions do not contain the degenerate fields in the Liouville sector, the index l here assumes summation of the "even" and of the "odd" conformal blocks in accordance with general OPE (10). Again, the normalization leaves only two basic combinations outside the conformal blocks,

$$\begin{aligned}
\mathcal{R}r_0(P) &= \mathbb{C}_{a,a}^{Q/2+iP}\mathbb{C}_{a,a}^{Q/2-iP},\\
\mathcal{R}r_1(P) &= \tilde{\mathbb{C}}_{a,a}^{Q/2+iP}\tilde{\mathbb{C}}_{a,a}^{Q/2-iP},
\end{aligned} \tag{45}$$

where we separate the factor $\mathcal{R}$ for convenience (it is independent of P) and the parameter a is related to the external conformal dimension (i. e. either b or $2b$). All other correlation functions in (39) are already not independent. They are expressed in terms of the same ingredients as in (40) and (44). For example,

$$\begin{aligned}
\langle \chi(z)\Phi(0)\chi(1)\Phi(\infty)\rangle &= c_k A_k^{(1)}(z)A_k^{(0)}(\bar{z}),\\
\langle \chi(z)\chi(0)\Phi(1)\Phi(\infty)\rangle &= c_k A_k^{(2)}(z)A_k^{(0)}(\bar{z}),\\
\langle \chi(z)\Phi(0)\Phi(1)\chi(\infty)\rangle &= c_k A_k^{(3)}(z)A_k^{(0)}(\bar{z}),
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\langle \bar{Y}(z)V(0)\bar{Y}(1)V(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P) B_l^{(0)}(P,z)\bar{B}_l^{(1)}(P,\bar{z}),\\
\langle \bar{Y}(z)\bar{Y}(0)V(1)V(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P) B_l^{(0)}(P,z)\bar{B}_l^{(2)}(P,\bar{z}),\\
\langle \bar{Y}(z)V(0)V(1)\bar{Y}(\infty)\rangle &= \mathcal{R}\int \frac{dP}{4\pi} r_l(P) B_l^{(0)}(P,z)\bar{B}_l^{(3)}(P,\bar{z}).
\end{aligned} \tag{47}$$

The remaining six correlation functions not written explicitly are obtained from (46) and (47) by complex conjugation. Using the introduced notation, we can rewrite the integrals under consideration in the compact forms

$$\mathcal{I}_4(b) = 2\mathcal{R}\int_{\mathbf{G}} d^2z \int \frac{dP}{4\pi}\sum_l r_l(P)\left[|B_l^{(1)}(P,z)|^2 + |B_l^{(2)}(P,z)|^2 + |B_l^{(3)}(P,z)|^2\right] \tag{48}$$

and

$$\begin{aligned}
\mathcal{I}_4(2b) = 2\mathcal{R}\int_{\mathbf{G}} d^2z \int \frac{dP}{4\pi}\sum_{k,l} c_k r_l(P)\Big[&|A_k^{(1)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(1)}(P,z)|^2\\
+ |A_k^{(2)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(2)}(P,z)|^2 &+ |A_k^{(3)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(3)}(P,z)|^2\Big].
\end{aligned} \tag{49}$$

Bulky expression (39) has a remarkably compact and clear structure in terms of the conformal blocks. We again note that in (48) and (49), we respectively assume the different values of external conformal dimensions in the Liouville sector Δ_b and Δ_{2b}.

5 The Modular Integral

It turns out efficient [23] to use elliptic transformations in the integration. We use the standard map

$$\tau = i\frac{K(1-x)}{K(x)}, \tag{50}$$

where the complete elliptic integral of the first kind is

$$K(x) = \frac{1}{2}\int_0^1 \frac{dt}{y} \tag{51}$$

and $y^2 = t(1-t)(1-xt)$. It can be verified that

$$dz = \pi z(1-z)\theta_3^4(q)d\tau, \tag{52}$$

where

$$q = e^{i\pi\tau} \tag{53}$$

and

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{54}$$

Integral (48) becomes

$$\begin{aligned} \mathcal{I}_4(b) = 2\pi^2\mathcal{R}\int_{-\infty}^{\infty}\frac{dP}{4\pi}\sum_l r_l(P)\Big[\int_{\mathbf{F}} |z(1-z)\theta_3^4(q)B_l^{(1)}(P,z)|^2 d^2\tau \\ + \int_{\mathbf{F}} |z(1-z)\theta_3^4(q)B_l^{(2)}(P,z)|^2 d^2\tau + \int_{\mathbf{F}} |z(1-z)\theta_3^4(q)B_l^{(3)}(P,z)|^2 d^2\tau\Big], \end{aligned} \tag{55}$$

where $\mathbf{F} = \{|\tau| > 1;\ |\mathrm{Re}\,\tau| < 1/2\}$. Similarly, for (49), we have

$$\begin{aligned} \mathcal{I}_4(2b) =& 2\pi^2\mathcal{R}\int_{-\infty}^{\infty}\frac{dP}{4\pi}\sum_{k,l} c_k r_l(P) \\ &\Big[\int_{\mathbf{F}} |z(1-z)\theta_3^4(q)(A_k^{(1)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(1)}(P,z))|^2 d^2\tau \\ &+ \int_{\mathbf{F}} |z(1-z)\theta_3^4(q)(A_k^{(2)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(2)}(P,z))|^2 d^2\tau \\ &+ \int_{\mathbf{F}} |z(1-z)\theta_3^4(q)(A_k^{(3)}(z)B_l^{(0)}(P,z) + A_k^{(0)}(z)B_l^{(3)}(P,z))|^2 d^2\tau\Big]. \end{aligned} \tag{56}$$

We now define the conformal blocks more explicitly,

$$\begin{aligned} &A_-^{(0)}(z) = F_{00}^{\mathrm{M}}(0,z), \quad A_0^{(0)}(z) = F_{01}^{\mathrm{M}}(b,z), \quad A_+^{(0)}(z) = F_{00}^{\mathrm{M}}(2b,z), \\ &A_-^{(1)}(z) = F_{11}^{\mathrm{M}}(0,z), \quad A_0^{(1)}(z) = F_{10}^{\mathrm{M}}(b,z), \quad A_+^{(1)}(z) = F_{11}^{\mathrm{M}}(2b,z), \\ &A_-^{(2)}(z) = F_{20}^{\mathrm{M}}(0,z), \quad A_0^{(2)}(z) = F_{21}^{\mathrm{M}}(b,z), \quad A_+^{(2)}(z) = F_{20}^{\mathrm{M}}(2b,z), \\ &A_-^{(3)}(z) = F_{31}^{\mathrm{M}}(0,z), \quad A_0^{(3)}(z) = F_{30}^{\mathrm{M}}(b,z), \quad A_+^{(3)}(z) = F_{31}^{\mathrm{M}}(2b,z). \end{aligned} \tag{57}$$

Here, the first argument of the symmetric conformal blocks defines the internal conformal dimension. The first lower index corresponds to one of the four basic types of conformal blocks we consider with respect to the set of external fields (see Appendix A); the second index is 0 if the corresponding block with the given internal conformal dimension is "even" and 1 if it is "odd". In the Liouville sector,

$$\begin{aligned} &B_0^{(0)}(z) = F_{00}^{\mathrm{L}}(P,z), \quad B_1^{(0)}(z) = F_{01}^{\mathrm{L}}(P,z), \\ &B_0^{(1)}(z) = F_{11}^{\mathrm{L}}(P,z), \quad B_1^{(1)}(z) = F_{10}^{\mathrm{L}}(P,z), \\ &B_0^{(2)}(z) = F_{20}^{\mathrm{L}}(P,z), \quad B_1^{(2)}(z) = F_{21}^{\mathrm{L}}(P,z), \\ &B_0^{(3)}(z) = F_{31}^{\mathrm{L}}(P,z), \quad B_1^{(3)}(z) = F_{30}^{\mathrm{L}}(P,z). \end{aligned} \tag{58}$$

For the first integral, the complicated expressions for the Liouville structure constants give

$$\mathcal{R} = \left(\pi\mu\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{(Q/b-4)} \Upsilon_b^6(b)\Upsilon_b^2(Q/2)\Upsilon_b^4(Q/2+b). \tag{59}$$

The special function $\Upsilon_b(x)$ is the standard element of the LFT (see [19] for the definition and properties). Explicitly,

$$\begin{aligned}\mathcal{R} = \left(\pi\mu\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{(Q/b-4)} \gamma^4(\frac{bQ}{2})b^{-4b^2}\times \\ \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{3(1-b^2)^2e^{-t}}{2b^2} - \frac{6\sinh^2((1-b^2)t/(4b))}{\sinh(t/(2b))\sinh(bt/2)}\right]\right\},\end{aligned} \tag{60}$$

where we use "shift" relations (see [19]) for the last upsilon function to extract the additional factor $(\gamma(\frac{bQ}{2}))^4$. This allows improving the accuracy of the comparison with the analytic result, which contains the same factor with a singularity at $b=1$. The P dependent parts are

$$r_0(P) = \frac{P^2\Upsilon_b(b\pm iP)\Upsilon_b(Q/2\pm iP)}{\Upsilon_b^2(b-Q/4\pm iP/2)\Upsilon_b^2(Q/4\pm iP/2)\Upsilon_b^2(b+Q/4\pm iP/2)\Upsilon_b^2(3Q/4\pm iP/2)}, \tag{61}$$

$$r_1(P) = \frac{4P^2\Upsilon_b(b\pm iP)\Upsilon_b(Q/2\pm iP)}{\Upsilon_b^2(3b/2-Q/4\pm iP/2)\Upsilon_b^4(b/2+Q/4\pm iP/2)\Upsilon_b^2(1/2/b+Q/4\pm iP/2)}, \tag{62}$$

where we again use the shift relations to move the arguments of all upsilon functions inside the strip $[0,Q]$ where the standard integral representation is applicable and we use the notation $\Upsilon_b(x\pm y) = \Upsilon_b(x+y)\Upsilon_b(x-y)$

$$\begin{aligned}r_0(P) = P^2\exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{(5-2b^2+5b^4)^2e^{-t}}{2b^2}+\right.\right. \qquad (63)\\ \left.\left.\frac{2\cos(\frac{Pt}{2})(\cosh(\frac{(1-3b^2)t}{4b})+2\cosh(\frac{(1+b^2)t}{4b})+\cosh(\frac{(3-b^2)t}{4b}))-\cos(Pt)(\cosh(\frac{(1-b^2)t}{2b})+1)-6}{\sinh(t/(2b))\sinh(bt/2)}\right]\right\},\end{aligned}$$

$$\begin{aligned}r_1(P) = 4P^2\exp\left\{\int_0^\infty \frac{dt}{t}\left[-\frac{5(1-b^2)^2e^{-t}}{2b^2}+\right.\right. \qquad (64)\\ \left.\left.\frac{2\cos(\frac{Pt}{2})(3\cosh(\frac{(1-b^2)t}{4b})+\cosh(\frac{3(1-b^2)t}{4b}))-\cos(Pt)(\cosh(\frac{(1-b^2)t}{2b})+1)-6}{\sinh(t/(2b))\sinh(bt/2)}\right]\right\}.\end{aligned}$$

For the second integral, we analogously find the explicit expressions

$$\begin{aligned}\mathcal{R} = \left(\pi\mu\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{(Q/b-8)} \gamma^4(b^2)\gamma^4(\frac{bQ}{2})\gamma^4(b^2+\frac{bQ}{2})b^{4(1-6b^2)}\times \\ \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{3(1-b^2)^2e^{-t}}{2b^2} - \frac{6\sinh^2((1-b^2)t/(4b))}{\sinh(t/(2b))\sinh(bt/2)}\right]\right\}\end{aligned} \tag{65}$$

and

$$r_0(P) = P^2 \exp\left\{ \int_0^\infty \frac{dt}{t} \left[-\frac{(5-18b^2+37b^4)e^{-t}}{2b^2} + \right.\right. \tag{66}$$

$$\left.\left. \frac{2\cos(\frac{Pt}{2})(\cosh(\frac{(1-7b^2)t}{4b}) + 2\cosh(\frac{(1+b^2)t}{4b}) + \cosh(\frac{(3-5b^2)t}{4b})) - \cos(Pt)(\cosh(\frac{(1-b^2)t}{2b}) + 1) - 6}{\sinh(t/(2b))\sinh(bt/2)} \right] \right\},$$

$$r_1(P) = 4P^2 \exp\left\{ \int_0^\infty \frac{dt}{t} \left[-\frac{(5-26b^2+37b^4)e^{-t}}{2b^2} + \right.\right. \tag{67}$$

$$\left.\left. \frac{2\cos(\frac{Pt}{2})(\cosh(\frac{(3-7b^2)t}{4b}) + 2\cosh(\frac{(1-b^2)t}{4b}) + \cosh(\frac{(1-5b^2)t}{4b})) - \cos(Pt)(\cosh(\frac{(1-b^2)t}{2b}) + 1) - 6}{\sinh(t/(2b))\sinh(bt/2)} \right] \right\}.$$

The conformal blocks can be evaluated effectively using a numerical algorithm based on the recurrence relations developed in [14, 24–28]. We do not use the elliptic recursion to construct the necessary conformal blocks here. It turns out that to attain a convincing accuracy of the results, we need to know a very few first terms in the x- (as well as q-) expansion of the conformal blocks. This information can be obtained directly starting from the very definition of the conformal blocks in terms of the chain vectors (see Appendix A). Nevertheless, the elliptic representation and especially the form of the prefactor (i.e., the Δ asymptotic of the conformal blocks; see [24]) is very useful. It allows verifying the explicit expressions for the correlation functions by checking the crossing symmetry requirement, and in particular, fixing up all the signs, which is difficult to do starting from general principles.

6 Numerics

With (55) and (57), calculating reduces to numerically integrating several integrals of the general form

$$\int_{\mathbf{F}} |z(1-z)\theta_3^4(q)\mathcal{F}_P(z)|^2 d^2\tau, \tag{68}$$

where $\mathcal{F}_P(z)$ is some Liouville conformal block like in (55) or some more complicated composite expression like in (57). The integrand can be developed as a double power series in q and $\bar{q}$ in accordance with the general expansion

$$z(1-z)\theta_3^4(q)\mathcal{F}_P(z) = (16q)^\alpha \sum_{r=0}^\infty b_r(P) q^r, \tag{69}$$

where α and the coefficients b_r are defined by the concrete choice of the function $\mathcal{F}_P(z)$. In each term, we can integrate in $\tau_2 = \operatorname{Im}\tau$ explicitly with the result in terms of the function

$$\Phi(A,r,l) = \int_{\mathbf{F}} d^2\tau\, |16q|^{2A}\, q^r \bar{q}^l = \frac{(16)^{2A}}{\pi(2A+r+l)} \int_{-1/2}^{1/2} \cos(\pi(r-l)x) e^{-\pi\sqrt{1-x^2}(2A+r+l)} dx. \tag{70}$$

For (55), we have the sum of six integrals of form (68), and we obtain the series

$$\mathcal{I}_4(b) = \pi\mathcal{R}\sum_{L=0}^{\infty}\left(A_L^{(1,e)} + A_L^{(1,o)} + A_L^{(2,e)} + A_L^{(2,o)} + A_L^{(3,e)} + A_L^{(3,o)}\right), \tag{71}$$

where in the last sum

$$\begin{aligned}
A_L^{(1,e)} &= \int_0^{\infty} r_2(P)dP\sum_{k=0}^{L} b_k^{(1,e)}(P)b_{L-k}^{(1,e)}(P)\Phi\left(P^2/2+Q^2/8-1/2,k,L-k\right),\\
A_L^{(1,o)} &= \int_0^{\infty} r_1(P)dP\sum_{k=0}^{L} b_k^{(1,o)}(P)b_{L-k}^{(1,o)}(P)\Phi\left(P^2/2+Q^2/8,k,L-k\right),\\
A_L^{(2,e)} &= \int_0^{\infty} r_2(P)dP\sum_{k=0}^{L} b_k^{(2,e)}(P)b_{L-k}^{(2,e)}(P)\Phi\left(P^2/2+Q^2/8-1/2,k,L-k\right),\\
A_L^{(2,o)} &= \int_0^{\infty} r_1(P)dP\sum_{k=0}^{L} b_k^{(2,o)}(P)b_{L-k}^{(2,o)}(P)\Phi\left(P^2/2+Q^2/8,k,L-k\right),\\
A_L^{(3,e)} &= \int_0^{\infty} r_1(P)dP\sum_{k=0}^{L} b_k^{(3,e)}(P)b_{L-k}^{(3,e)}(P)\Phi\left(P^2/2+Q^2/8-1,k,L-k\right),\\
A_L^{(3,o)} &= \int_0^{\infty} r_2(P)dP\sum_{k=0}^{L} b_k^{(3,o)}(P)b_{L-k}^{(3,o)}(P)\Phi\left(P^2/2+Q^2/8-1/2,k,L-k\right).
\end{aligned} \tag{72}$$

Each term in (72) is suppressed by a factor $\max_{\mathbf{F}}|q|^{2L}$, and the series in L converges very rapidly in practice. We found that it suffices to sum up to $L=4$ to reach the three- to four-digit precision (see Table 1). In the case $(1,3)$, we numerically integrate in the same way although we now have 18 integrals of form (68). In this case summing up to $L=4$ we were able to reach the two- to three-digit precision (see Table 2). In Figs. 1 and 2, the results of numerically evaluating integrals (29) and (31) are shown as circles while the lines correspond to the exact result.

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b	$\Sigma^{(1,1)}(b)$ num.	$\Sigma^{(1,1)}(b)$ exact
0.999	0.9959	0.9960
0.95	0.8049	0.8050
0.85	0.4450	0.4450
0.80	0.2799	0.2800
$1/\sqrt{2}$	0.0001	0
0.65	0.1555	0.1550
0.60	0.2877	0.2800

Table 1: Numerical data for $\Sigma^{(1,1)}(b)$ at $\mu = 1$.

b	$\Sigma^{(1,3)}(b)$ num.	$\Sigma^{(1,3)}(b)$ exact
0.71	0.0309	0.0246
0.69	-0.1377	-0.1434
0.67	-0.3030	-0.3066
0.65	-0.4623	-0.4650
0.63	-0.6159	-0.6186
0.61	-0.7644	-0.7674
0.59	-0.9096	-0.9114
0.57	-0.9699	-0.9747
0.55	-0.9060	-0.9075
0.53	-0.8409	-0.8427
0.51	-0.7791	-0.7803
0.49	-0.7197	-0.7203
0.47	-0.6621	-0.6627
0.45	-0.6069	-0.6075
0.43	-0.3342	-0.3282
0.41	-0.0384	-0.0258
0.39	0.1827	0.2622

Table 2: Numerical data for $\Sigma^{(1,3)}(b)$ at $\mu = 1$.

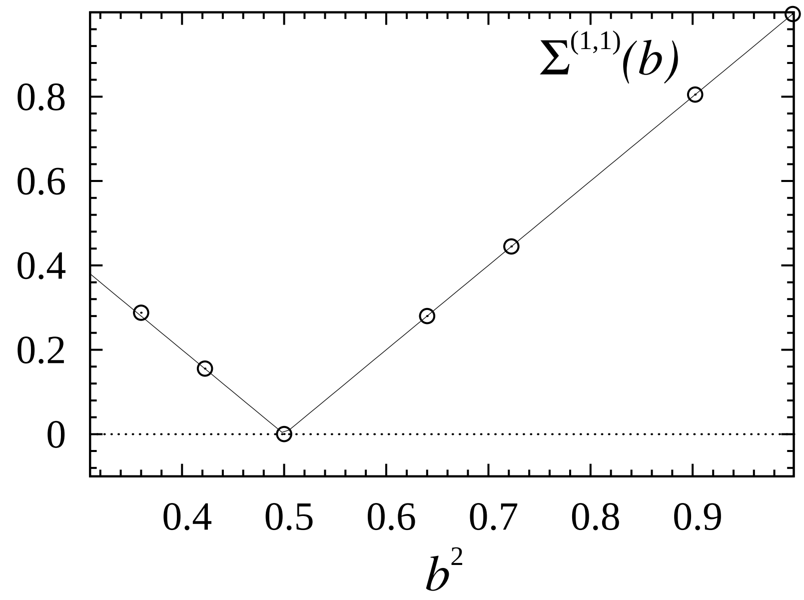

Figure 1: Direct numerical evaluation of reduced integral (29) (circles) versus the exact formula (continuous line).

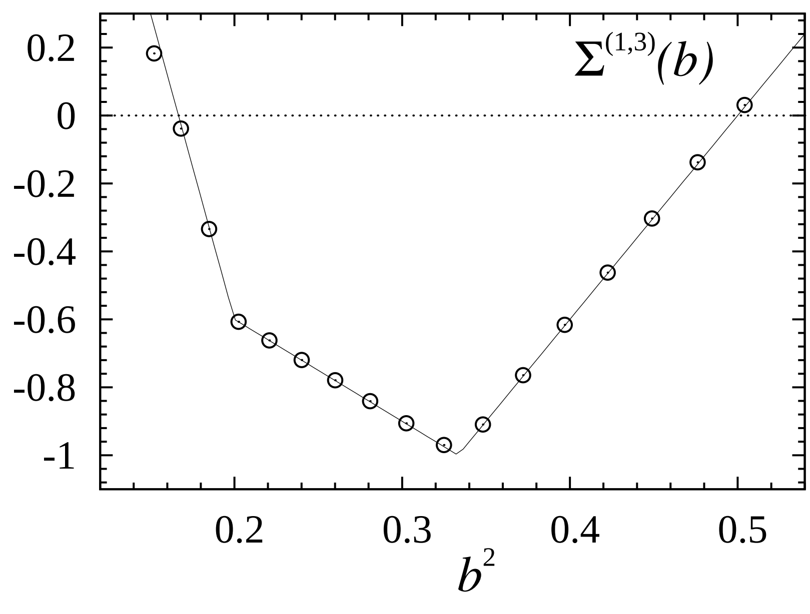

Figure 2: Direct numerical evaluation of reduced integral (31) (circles) versus the exact formula (continuous line).

Appendix A. Conformal Blocks and the Chain Vectors

For definiteness, we use the notation for the Liouville sector, though the results concerning the conformal block are universal (i.e., are independent of the sector). Schematically, the contribution of the given conformal family in the four basic OPE can be written as

$$
\begin{aligned}
V_1(z)V_2(0) &= z^{\Delta-\Delta_1-\Delta_2}\sum_N z^N|N\rangle_{12},\\
W_1(z)V_2(0) &= z^{\Delta-\Delta_1-\Delta_2-1/2}\sum_N z^N\widetilde{|N\rangle}_{12},\\
V_1(z)W_2(0) &= z^{\Delta-\Delta_1-\Delta_2-1/2}\sum_N z^N\widetilde{\widetilde{|N\rangle}}_{12},\\
W_1(z)W_2(0) &= z^{\Delta-\Delta_1-\Delta_2-1}\sum_N z^N\widetilde{\widetilde{\widetilde{|N\rangle}}}_{12},
\end{aligned}
\tag{73}
$$

where the so-called chain vectors $|N\rangle, \widetilde{|N\rangle}, \widetilde{\widetilde{|N\rangle}}, \widetilde{\widetilde{\widetilde{|N\rangle}}}$ (with positive integer or half-integer N) are the Nth-level descendent contribution of the intermediate state with the conformal dimension Δ appearing in the given operator product expansion. The chain vectors are completely determined by the superconformal symmetry. Namely, the superconformal constraints lead to the recurrence relations

$$
\begin{cases}
G_k|N\rangle_{12} = \widetilde{|N-k\rangle}_{12},\\
G_k\widetilde{|N\rangle}_{12} = [\Delta + 2k\Delta_1 - \Delta_2 + N - k]|N-k\rangle_{12}
\end{cases}
\tag{74}
$$

for $k > 0$. And

$$
\begin{cases}
G_k\widetilde{\widetilde{|N\rangle}}_{12} = \widetilde{\widetilde{\widetilde{|N-k\rangle}}}_{12} + 2\Delta_2\delta_{k,1/2}|N-k\rangle_{12},\\
G_k\widetilde{\widetilde{\widetilde{|N\rangle}}}_{12} = [\Delta + 2k\Delta_1 - (\Delta_2+1/2) + N - k]\widetilde{\widetilde{|N-k\rangle}}_{12} - 2\Delta_2\delta_{k,1/2}\widetilde{|N-k\rangle}_{12}
\end{cases}
\tag{75}
$$

for $k > 0$. The normalization of the chain vectors chosen in this text is determined by the requirements

$$
|0\rangle = 1, \quad \widetilde{|0\rangle} = 1, \quad \widetilde{\widetilde{|0\rangle}} = -1, \quad \widetilde{\widetilde{\widetilde{|0\rangle}}} = (\Delta - \Delta_1 - \Delta_2). \tag{76}
$$

Relations (74) and (75) are equivalent to the linear problem for the coefficients determining the chain vectors in terms of the Virasoro basis vectors of the same level. These systems can be solved numerically up to a rather high level.

The necessary s-channel superconformal blocks are defined via the expansions

$$\mathcal{F}_{\mathrm{e,o}}\left(\begin{array}{cc} a_1 & a_3 \\ a_2 & a_4 \end{array}\middle|\Delta\middle| z\right) = z^{\Delta-\Delta_1-\Delta_2}\sum_{N\in\mathbb{Z},\mathbb{Z}/2} z^N{}_{12}\langle N|N\rangle_{34}, \tag{77}$$

$$\mathcal{F}_{\mathrm{e,o}}\left(\begin{array}{cc} \hat{a}_1 & \hat{a}_3 \\ a_2 & a_4 \end{array}\middle|\Delta\middle| z\right) = z^{\Delta-\Delta_1-\Delta_2-1/2}\sum_{N\in\mathbb{Z},\mathbb{Z}/2} z^N{}_{12}\langle \widetilde{N}|\widetilde{N}\rangle_{34}, \tag{78}$$

$$\mathcal{F}_{\mathrm{e,o}}\left(\begin{array}{cc} \hat{a}_1 & a_3 \\ \hat{a}_2 & a_4 \end{array}\middle|\Delta\middle| z\right) = z^{\Delta-\Delta_1-\Delta_2-1}\sum_{N\in\mathbb{Z},\mathbb{Z}/2} z^N{}_{12}\langle \widetilde{\widetilde{\widetilde{N}}}|N\rangle_{34}, \tag{79}$$

$$\mathcal{F}_{\mathrm{e,o}}\left(\begin{array}{cc} \hat{a}_1 & a_3 \\ a_2 & \hat{a}_4 \end{array}\middle|\Delta\middle| z\right) = z^{\Delta-\Delta_1-\Delta_2-1/2}\sum_{N\in\mathbb{Z},\mathbb{Z}/2} z^N{}_{12}\langle \widetilde{N}|\widetilde{\widetilde{N}}\rangle_{34}. \tag{80}$$

In the main text, we use the brief notation

$$F_{00}(\Delta,z) = \mathcal{F}_{\mathrm{e}}\left(\begin{array}{cc} a & a \\ a & a \end{array}\middle|\Delta\middle| z\right), \quad F_{01}(\Delta,z) = \mathcal{F}_{0}\left(\begin{array}{cc} a & a \\ a & a \end{array}\middle|\Delta\middle| z\right), \tag{81}$$

$$F_{10}(\Delta,z) = \mathcal{F}_{\mathrm{e}}\left(\begin{array}{cc} \hat{a} & \hat{a} \\ a & a \end{array}\middle|\Delta\middle| z\right), \quad F_{11}(\Delta,z) = \mathcal{F}_{0}\left(\begin{array}{cc} \hat{a} & \hat{a} \\ a & a \end{array}\middle|\Delta\middle| z\right), \tag{82}$$

$$F_{20}(\Delta,z) = \mathcal{F}_{\mathrm{e}}\left(\begin{array}{cc} \hat{a} & a \\ \hat{a} & a \end{array}\middle|\Delta\middle| z\right), \quad F_{21}(\Delta,z) = \mathcal{F}_{0}\left(\begin{array}{cc} \hat{a} & a \\ \hat{a} & a \end{array}\middle|\Delta\middle| z\right), \tag{83}$$

$$F_{30}(\Delta,z) = \mathcal{F}_{\mathrm{e}}\left(\begin{array}{cc} \hat{a} & a \\ a & \hat{a} \end{array}\middle|\Delta\middle| z\right), \quad F_{31}(\Delta,z) = \mathcal{F}_{0}\left(\begin{array}{cc} \hat{a} & a \\ a & \hat{a} \end{array}\middle|\Delta\middle| z\right). \tag{84}$$

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